SHORT NOTE ON THE CONVOLUTION OF BINOMIAL COEFFICIENTS

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Abstract. We know [1] that, for every non-negative integer numbers n, i, j and for every real number ℓ,

$$\sum_{i+j=n} \binom{2\,i-\ell}{i}\binom{2\,j+\ell}{j} = \sum_{i+j=n} \binom{2\,i}{i}\binom{2\,j}{j}, \tag{1}$$

which is well-known to be 4^n. We extend this result by proving that, indeed,

$$\sum_{i+j=n} \binom{a\,i+k-\ell}{i}\binom{a\,j+\ell}{j} = \sum_{i+j=n} \binom{a\,i+k}{i}\binom{a\,j}{j} \tag{2}$$

for every integer a and for every real k, and present new expressions for this value.

We consider the sequence $\left\{\binom{a\,n}{n}\right\}_{n=0}^{\infty}$, where a is any integer number, negative, zero or positive, and take the convolution of this sequence with itself, defined by $P_a(n) = \sum_{i+j=n}\binom{a\,i}{i}\binom{a\,j}{j}$.

When $a = 2$, the former is sequence A000984 of [2], the central binomial coefficients, and the latter is sequence $A000302$ of [2], the powers of 4. In fact (cf. [1]), this can be proved directly using (1), and then the inclusion-exclusion principle. Note that

$$2\,P_2(n) = 2^{2n+1} = \sum_{i=0}^{2n+1}\binom{2n+1}{i} = 2\sum_{i=0}^{n}\binom{2n+1}{i}. \tag{3}$$

For another identity, define as usual $[n] = \{1, \ldots, n\}$ for any natural number n, and consider the collection of the subsets of $[2n]$ with more than n elements with the same $(n+1)$-th element, say p. Note that $p = n+1+i$ for some $i = 0, \ldots, n-1$ and that there are $\binom{n+i}{n}\,2^{n-i-1}$ subsets in the collection. It follows that the number of all subsets of $[2n]$ is

$$P_2(n) = 2^{2n} = 2\sum_{i=0}^{n-1} 2^{n-i-1}\binom{n+i}{i} + \binom{2n}{n} = \sum_{i=0}^{n} 2^{n-i}\binom{n+i}{i}. \tag{4}$$

We generalize these identities, namely (1), (3) and (4). When $a = 3$ and $a = 4$, we have sequences $A006256$ and $A078995$ of [2], and no such simple formulas for $P_3(n)$ and $P_4(n)$ are known as in case $a = 2$. For these sequences, we obtain, for every real ℓ,

$$\sum_{i+j=n}\binom{3i}{i}\binom{3j}{j} = \sum_{i+j=n} 2^i\binom{3n+1}{j} = \sum_{i+j=n} 3^i\binom{2n+j}{j} = \sum_{i+j=n}\binom{3i-\ell}{i}\binom{3j+\ell}{j}$$

$$\sum_{i+j=n}\binom{4i}{i}\binom{4j}{j} = \sum_{i+j=n} 3^i\binom{4n+1}{j} = \sum_{i+j=n} 4^i\binom{3n+j}{j} = \sum_{i+j=n}\binom{4i-\ell}{i}\binom{4j+\ell}{j}$$

More generally we obtain the following theorem.

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Theorem 1. *For every non-negative integer numbers i, j and n, and for every real numbers k and ℓ,*

$$\sum_{i+j=n} \binom{a\,i+k-\ell}{i}\binom{a\,j+\ell}{j} = \sum_{i+j=n} \binom{a\,i+k}{i}\binom{a\,j}{j}$$

$$= \sum_{i=0}^{n} (a-1)^{n-i} \binom{a\,n+k+1}{i} \tag{5}$$

$$= \sum_{i=0}^{n} a^{n-i} \binom{(a-1)n+k+i}{i} \tag{6}$$

where we take $0^0 = 1$.

For the proof of this theorem we need some technical results.

Lemma 2. *Let, for any real ℓ and integers a and n such that $n \geq 0$,*

$$S_{a,\ell}(n) \;=\; \sum_{i=0}^{n} (-1)^i \binom{\ell-(a-1)i}{i}\binom{\ell-a\,i}{n-i}$$

Then

$$\sum_{i=0}^{n} \binom{n}{p} S_{a,\ell}(p) \;=\; S_{a+1,\ell+n}(n).$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{p} S_{a,\ell}(p) &= \sum_{i=0}^{n} \left[(-1)^i \binom{\ell-(a-1)i}{i} \sum_{p=i}^{n} \binom{\ell-a\,i}{p-i}\binom{n}{p} \right] \\
&= \sum_{i=0}^{n} \left[(-1)^i \binom{\ell-(a-1)i}{i} \sum_{p=i}^{n} \binom{\ell-a\,i}{\ell-(a-1)i-p}\binom{n}{p} \right] \\
&= \sum_{i=0}^{n} (-1)^i \binom{\ell-(a-1)i}{i}\binom{\ell+n-a\,i}{\ell-(a-1)\,i} \\
&= \sum_{i=0}^{n} (-1)^i \binom{(\ell+n)-a\,i}{i\;,\;n-i\;,\;\ell-ai} \\
&= \sum_{i=0}^{n} (-1)^i \binom{(\ell+n)-a\,i}{i}\binom{(\ell+n)-(a+1)\,i}{n-\,i}
\end{aligned}$$

where we use Vandermonde's convolution in the third equality. □

Lemma 3. *With the notation of the previous lemma,*

$$S_{a,\ell}(n) \;=\; (a-1)^n.$$

Proof. First note that we may assume that ℓ is a natural number, since $S_{a,\ell}(n)$ is a polynomial in ℓ, and thus is constant. Now, suppose that $S_{a,\ell}(p) = x^p$ for some numbers a, ℓ, p and x. Then, from Lemma 2 it follows that $S_{a+1,\ell+n}(n) = (1+x)^n$. Hence, all we must prove is that $S_{a,\ell}(n) = 0$ when $a = 1$ and $\ell \in \mathbb{N}$.

For this purpose, define $\mathcal{A} = \mathcal{A}_\varnothing$ as the set of n-subsets of the set $[\ell] = \{1, 2, \ldots, \ell\}$ and, for every non-empty subset T of $[\ell]$, $\mathcal{A}_T = \big\{ A \in \mathcal{A} \mid A \cap T = \varnothing \big\}$. Now, the result follows immediately from the inclusion-exclusion principle applied to this family. □

Lemma 4. *Let s and t be positive integers. Then*

$$\binom{s+t+1}{j} = \sum_{i=0}^{j} \binom{s-i}{s-j}\binom{t+i}{i}.$$

Proof. Given a subset S of $[n]$ with k elements and $p \in [n] \setminus S$, let $\mathrm{Bef}_p(S) = S \cap [p-1]$ and $\mathrm{Aft}_p(S) = \{t \in [n-p] \mid t+p \in S\}$.

Now, let A be a subset of $[s+t+1]$ with j elements and $p(A)$ be the $s-j+1$ smallest element of $[s+t+1]$ which is not in A. In other words, $\#\{x \in A \mid x < p(A)\} = j-i$ and $\#\{x \in A \mid x > p(A)\} = i$. One can easily see that the mapping

$$\begin{array}{rccc} \varphi: & \mathcal{P}_j([s+t+1]) & \to & \bigcup\limits_{0\le i\le j} \mathcal{P}_{j-i}([s-i]) \times \mathcal{P}_i([t+i]) \\ & A & \mapsto & (\mathrm{Bef}_{p(A)}(A), \mathrm{Aft}_{p(A)}(A)) \end{array}$$

is a bijection, with inverse given by $\psi(B,C) = B \cup \{c + \#C \mid c \in C\}$, and the union is disjoint. $\square$

Proof of Theorem 1. Let $\mathfrak{S} = \sum_{i+j=n} \binom{a\,i+k-\ell}{i}\binom{a\,j+\ell}{j} = \sum_{i+j=n} (-1)^i \binom{\ell-k'-(a-1)i}{i}\binom{a\,n+\ell-a\,i}{j}$, with $k' = k+1$. Then, by Vandermonde's convolution,

$$\begin{aligned} \mathfrak{S} &= \sum_{i+j=n}\left[(-1)^i\binom{\ell-k'-(a-1)i}{i}\sum_{p+m=j}\binom{a\,n+k'}{p}\binom{\ell-k'-a\,i}{m}\right] \\ &= \sum_{p=0}^{n}\left[\binom{a\,n+k'}{p}\sum_{i+m=n-p}(-1)^i\binom{\ell-k'-(a-1)i}{i}\binom{\ell-k'-a\,i}{m}\right] \end{aligned}$$

Now, (5) follows immediately from Lemma 3 and (6) from Lemma 4. $\square$

We end this article with a new result that, when we represent by $\left(\!\binom{n}{k}\!\right)$ the number $\binom{n+k-1}{k}$ of k-multisets of elements of an n-set, can be formulated in the following elegant terms.

Theorem 5. *For every real ℓ and integers a, n, i, j such that $n, i, j \geq 0$,*

$$\sum_{i+j=n} (-1)^i \left(\!\binom{\ell-a\,i}{i}\!\right)\binom{\ell-a\,i}{j} = a(a-1)^{n-1}.$$

Proof. By Pascal's rule,

$$\begin{aligned} \sum_{i+j=n}(-1)^i\binom{\ell-1-(a-1)i}{i}\binom{\ell-a\,i}{j} =& \sum_{i=0}^{n}(-1)^i\binom{\ell-(a-1)i}{i}\binom{\ell-a\,i}{n-i} \\ & -\sum_{i=1}^{n}(-1)^i\binom{\ell-(a-1)i-1}{i-1}\binom{\ell-a\,i}{n-i} \\ =& S_{a,\ell}(n) + S_{a,\ell-a}(n-1) \end{aligned}$$

$\square$

Problem 6. *Give a full combinatorial proof of Theorem 5.*

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References

[1] Rui Duarte and António Guedes de Oliveira, New developments of an old identity, manuscript `arXiv:1203.5424`, *submitted.*

[2] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at `http://oeis.org`, 2011.

[3] Richard Stanley, *Enumerative Combinatorics*, Vol. 1, Cambridge Studies in Advanced Mathematics **49** Cambridge University Press, Cambridge, 1997.

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